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The treatment of the locking phenomenon for a general class of variational inequalities

Leila Slimane[a,*], Yves Renard[b]

[a] Université de Moncton, Campus de Shippagan, 218, Boulevard J.D. Gauthier, Shippagan, Canada E8S 1P6
[b] MIP, INSAT, Complexe scientifique de Rangueil, 31077 Toulouse, France

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Abstract

We present the analysis of a class of variational inequalities depending on a small nonnegative parameter in a singular way, for which direct numerical approximation yield a numerical locking phenomenon. It consists in extending some robust approaches to variational inequalities, mainly, conforming and nonconforming methods. We give general sufficient conditions on the discrete problem insuring a uniform convergence relatively to the small parameter, and consequently avoiding numerical locking.

Keywords: Numerical locking; Variational inequalities; Signorini problems; Stiff transmission; Nonconforming methods

1. Introduction

Some unilateral problems depend on a small parameter ε: stiff transmission with Signorini boundary conditions, thin structures, unilateral problems in nearly incompressible elasticity. The common difficulty in these problems is due to their singular dependence on this parameter ε when it vanishes. The direct numerical approximation can yield a numerical locking phenomenon, which generally consists in loss of meaning of the numerical results, when the parameter is relatively smaller than the discretization parameter. From a mathematical point of view, the locking corresponds to the absence of the uniform convergence with respect to ε.

The locking phenomenon has been known for a long time in the world of engineering and has been widely studied in the last few years. Babuška and Sûri develop precise mathematical definitions for locking and robustness, as well as their quantitative characterizations [2]. They suggest conditions

* Corresponding author.
E-mail address: lslimane@cums.ca (L. Slimane).

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which allow to avoid numerical locking in the linear case when we use standard conforming finite element methods.

We note that in the linear case many methods have been proposed in order to avoid this phenomenon: mixed methods [7,23], nonconforming methods of lowest degree [5,9], conforming methods with lowest degree and with a particular choice of the mesh [23], p and hp methods [2], and asymptotic analysis relatively to the small parameter [18,23].

In this paper, we treat a large class of variational inequalities depending on a small parameter ε. We try to give sufficient conditions in order to ensure uniform convergence of the discrete solution. This generalizes the conditions established for the variational equations in [10] in the conforming case, and by Papaghuic and Thomas in the nonconforming case [9].

This paper is organized as follows. In Section 2, we start by giving the general setting of the class of the considered problems for which asymptotic analysis of the problem relatively to the small parameter is given. In Section 3, we present sufficient conditions on the approximation method in order to ensure the uniform convergence. Finally, in Section 4, the general result is applied to a Signorini problem with a stiff transmission, and numerical experiments are presented.

2. Continuous problem and asymptotic analysis

Let V be a real Hilbert space equipped with the norm $\|.\|$ and $K \subseteq V$ a nonempty closed convex subset. We consider two symmetric continuous bilinear forms a_0, a_1 defined on $V \times V$ and L a continuous linear form on V. Let $\varepsilon \in]0,1]$ be a small parameter. We define $a^\varepsilon : V \times V \to \mathbb{R}$ by

$$a^\varepsilon = a_1 + \varepsilon^{-1} a_0.$$

We will consider the following hypotheses:

(H1) The bilinear form a^1 (a^ε with $\varepsilon = 1$) is coercive

$$\exists \alpha > 0 \quad \text{such that} \quad \forall v \in V, \quad a^1(v,v) \geqslant \alpha \|v\|^2.$$

(H2) The bilinear form a_0 is positive and its kernel

$$G = \{w \in V;\ a_0(w,v) = 0, \quad \forall v \in V\}$$

is not reduced to $\{0\}$; additionally, the intersection $K \cap G$ is nonempty.

Since a_0 is positive and symmetric, we have $|a_o(w,v)| \leqslant a_o(w,w)^{1/2} a_0(v,v)^{1/2}$ in such a way that G can be written as follows:

$$G = \{w \in V;\ a_0(w,w) = 0\}.$$

In particular, these hypotheses imply that the forms a^ε, for $\varepsilon \in]0,1]$, are uniformly coercive, and a_1 is coercive on G

$$a_1(w,w) \geqslant \alpha \|w\|^2 \quad \forall w \in G.$$

We now consider the following problem:

$$u^\varepsilon \in K, \quad a^\varepsilon(u^\varepsilon, v - u^\varepsilon) \geqslant L(v - u^\varepsilon) \quad \forall v \in K. \tag{1}$$

The Stampacchia theorem [6] ensures that Problem (1) admits a unique solution for all $\varepsilon \in]0,1]$. The result holds also for the following problem:

$$u^0 \in K_0, \quad a_1(u^0, w - u^0) \geqslant L(w - u^0) \quad \forall w \in K_0, \tag{2}$$

where the closed convex subset K_0 is given by

$$K_0 = \{w \in K;\ a_0(w, v) = 0, \quad \forall v \in V\} = K \cap G.$$

Theorem 1. *Under hypotheses* (H1) *and* (H2), *the solution* u^ε *to Problem* (1) *converges strongly in* V, *when* $\varepsilon \to 0$, *to* u^0 *the unique solution to Problem* (2).

Proof. This result can be obtained in a similar manner as the one given in [10]. The proof can be splited into three steps.

A priori estimate and weak limit: Let v_0 be a fixed element of K_0. We have then

$$a^\varepsilon(u^\varepsilon, v_0 - u^\varepsilon) \geqslant L(v_0 - u^\varepsilon).$$

Hence

$$a^\varepsilon(u^\varepsilon - v_0, u^\varepsilon - v_0) \leqslant L(u^\varepsilon - v_0) - a^\varepsilon(v_0, u^\varepsilon - v_0). \tag{3}$$

The coercivity of a^ε gives

$$\alpha\|u^\varepsilon - v_0\|^2 \leqslant L(u^\varepsilon - v_0) - a_1(v_0, u^\varepsilon - v_0)$$

which leads to

$$\|u^\varepsilon - v_0\| \leqslant C. \tag{4}$$

Therefore, we deduce that the family $(u^\varepsilon)_{\varepsilon>0}$ is bounded in V. Consequently, there exists a subsequence, still denoted by $(u^\varepsilon)_{\varepsilon>0}$, which converges weakly in V to an element $w^0 \in K$.

Properties of the weak limit: It follows from (3) and (4) that

$$a_0(u^\varepsilon, u^\varepsilon) \leqslant C\varepsilon,$$

where C is a positive constant independent of ε. Using the lower semi-continuity of a_0, we deduce that $a_0(w^0, w^0) = 0$ which means that $w^0 \in K_0$. By choosing $w \in K_0$ in (1) as a test function and taking into account the fact that $\varepsilon^{-1} a_0(u^\varepsilon, -u^\varepsilon) \leqslant 0$, we get

$$a_1(u^\varepsilon, w - u^\varepsilon) \geqslant L(w - u^\varepsilon) \quad \forall w \in K_0.$$

Finally, the passage to the limit gives

$$a_1(w^0, w - w^0) \geqslant L(w - w^0) \quad \forall w \in K_0$$

which is nothing but (2). The uniqueness of the solution to this last variational inequality implies $w^0 = u^0$ and consequently, the whole sequence $(u^\varepsilon)_\varepsilon$ converges to u^0.

Strong convergence: We have

$$\alpha\|u^\varepsilon - u^0\|^2 \leqslant a^\varepsilon(u^\varepsilon - u^0, u^\varepsilon - u^0) = a^\varepsilon(u^\varepsilon, u^\varepsilon - u^0) - a^\varepsilon(u^0, u^\varepsilon - u^0).$$

Taking into account (1) and the fact that u^0 is in K_0, we obtain

$$a^\varepsilon(u^\varepsilon - u^0, u^\varepsilon - u^0) \leqslant L(u^\varepsilon - u^0) - a_1(u^0, u^\varepsilon - u^0).$$

Then, by a passage to the limit we get

$$\lim_{\varepsilon\to 0} a^{\varepsilon}(u^{\varepsilon}-u^0, u^{\varepsilon}-u^0) = 0. \tag{5}$$

By using again the uniform coercivity of a^{ε}, we obtain

$$\lim_{\varepsilon\to 0} \|u^{\varepsilon}-u^0\| = 0,$$

i.e., the strong convergence of $(u^{\varepsilon})_{\varepsilon>0}$ to u^0. $\square$

3. Discrete problem and uniform convergence

Our aim here is to construct an approximation u_h^{ε} of u^{ε} by using the standard Galerkin's method such that the sequence u_h^{ε} approaches correctly u^{ε} without a locking phenomenon when ε is closed to 0.

3.1. The discrete problem

Let H_h be a Hilbert space, equipped with the norm $\|.\|_h$, which depends on the discretization parameter $h>0$. Let $(V_h)_{h>0}$ be a family of finite-dimensional subspaces of H_h. In all of the following, the space V is supposed to be a closed subspace of H_h, and

$$\forall v \in V, \quad \|v\|_h = \|v\|.$$

We will consider a general case which contains the conforming case ($V_h \subset V$) and the nonconforming case ($V_h \not\subset V$). For this, we suppose that the bilinear forms a_0, a_1 and the linear form L are defined and continuous on $V + V_h$ with respect to the norm $\|.\|_h$ and with continuity constants independent of h. We now consider the following hypotheses:

(H0$^{\mathrm{nc}}$) For every bounded sequence $v_h \in V_h$ there exists a subsequence which converges weakly in H_h to an element $v \in V$.

(H1$^{\mathrm{nc}}$) The bilinear form $a^1 = a_1 + a_0$ is uniformly coercive on V_h

$$\exists \beta > 0 \quad \text{such that} \quad a^{\varepsilon}(v_h, v_h) \geqslant \beta \|v_h\|_h^2 \quad \forall v_h \in V_h \;\; \forall \varepsilon \in]0,1] \;\; \forall h > 0.$$

(H2$^{\mathrm{nc}}$) The bilinear form a_0 is positive.

Let K_h be a nonempty closed convex subset of V_h such that $K_{0h} = K_h \cap G_h$ is nonempty, where G_h is the kernel of a_0 on V_h. For each fixed $\varepsilon \in]0,1]$, we approach inequality (1) by the following problem:

$$u_h^{\varepsilon} \in K_h, \quad a^{\varepsilon}(u_h^{\varepsilon}, v_h - u_h^{\varepsilon}) \geqslant L(v_h - u_h^{\varepsilon}) \;\; \forall v_h \in K_h. \tag{6}$$

For the same reason as for the continuous case, Problem (6) has a unique solution for all $h>0$.

The discrete problem obtained by the Galerkin's method from Problem (2) is given by

$$u_h^0 \in K_{0h}, \quad a_1(u_h^0, w_h - u_h^0) \geqslant L(w_h - u_h^0) \quad \forall w_h \in K_{0h}. \tag{7}$$

We note that the bilinear form a_1 is coercive on G_h with coercivity constant independent of h. So we deduce that Problem (7) has a unique solution for all $h>0$.

Now, we study the following limits:

$$\lim_{\varepsilon\to 0} u_h^\varepsilon \quad \text{with } h \text{ fixed},$$

$$\lim_{h\to 0} u_h^\varepsilon \quad \text{with } \varepsilon \text{ fixed}.$$

Concerning the first passage to the limit, in the same way as for the continuous case we can establish the following result.

Proposition 1. *Let $h > 0$ be given. The solution u_h^ε to Problem (6) converges in V_h to the solution u_h^0 to Problem (7) when $\varepsilon \to 0$.*

In order to study the second limit, we start by introducing a new notion of the approximation of a family of subsets well adapted in the nonconforming case. Then, we establish an error estimate in this framework.

Definition 1. We say that a family of subsets $\mathscr{C}_h$ of V_h approaches $\mathscr{C} \subseteq V$ if the following conditions are satisfied:

1. Approximation property

$$\forall v \in \mathscr{C}, \quad \lim_{h\to 0} \inf_{v_h \in \mathscr{C}_h} \|v - v_h\|_h = 0. \tag{8}$$

2. Stability by passage to the weak limit

$$v_h \in \mathscr{C}_h \text{ and } v_h \rightharpoonup v \text{ weakly in } H_h \text{ implies } v \in \mathscr{C}. \tag{9}$$

Remark 1. In the conforming case, this definition is given in [21].

3.2. Error estimate

Our aim now is to give an error estimate in a more general case than the one in the problem given in Section 2. We consider the bilinear form a which is defined on $(V + V_h) \times (V + V_h)$ and satisfies

$$\begin{aligned} &\exists M > 0, \quad a(u,v) \leqslant M\|u\|_h\|v\|_h \ \forall u,v \in V + V_h,\\ &\exists \alpha_1 > 0, \quad a(v,v) \geqslant \alpha_1\|v\|^2 \ \forall v \in V,\\ &\exists \alpha_2 > 0, \quad a(v_h,v_h) \geqslant \alpha_2\|v_h\|_h^2 \ \forall h > 0 \ \forall v_h \in V_h. \end{aligned} \tag{10}$$

Finally, we give a linear continuous form f defined on $V + V_h$ with a continuity constant independent of h.

The problem under study is as follows:

$$u \in K, \quad a(u, v-u) \geqslant f(v-u) \ \forall v \in K, \tag{11}$$

whose discrete formulation is given by

$$u_h \in K_h, \quad a(u_h, v_h - u_h) \geqslant f(v_h - u_h) \ \forall v_h \in K_h. \tag{12}$$

Each of problems (11) and (12) has a unique solution.

Theorem 2. *Under hypothesis* (10), *if* u (*resp.* u_h) *is the solution to* (11) (*resp.* (12)) *we have the following estimate*:

$$\begin{aligned}\|u - u_h\|_h^2 \leqslant C(&\inf_{v_h \in K_h} \{\|u - v_h\|_h^2 + a(u, v_h - u) - f(v_h - u)\} \\ &+ \inf_{v \in K} \{a(u, v - u_h) - f(v - u_h)\}),\end{aligned} \tag{13}$$

where C *is a positive constant independent of the parameter* h, *and which depends linearly on the constants* M *and* $1/\alpha_2$.

Remark 2. The error estimate (13) generalizes to the nonconforming case ($V_h \not\subset V$) the one of the Falk lemma (cf. [13,17]) established in the conforming case ($V_h \subset V$).

Proof. The difficulty resides in the fact that a is not necessarily coercive on $V + V_h$. Let v_h be an arbitrary element of K_h, then we have

$$\alpha_2 \|v_h - u_h\|_h^2 \leqslant a(v_h - u_h, v_h - u_h). \tag{14}$$

By developing, we get

$$\begin{aligned}a(v_h - u_h, v_h - u_h) = {}& a(v_h - u, v_h - u_h) + a(u - u_h, u - u_h) \\ &+ a(u - u_h, v_h - u).\end{aligned} \tag{15}$$

The estimate on the second term of the right-hand side is obtained following a classical method. On the one hand, we have for all $v \in K$;

$$\begin{aligned}a(u, u - u_h) &= a(u, u - v) + a(u, v - u_h), \\ &\leqslant -f(v - u) + a(u, v - u_h),\end{aligned}$$

which is equivalent to

$$a(u, u - u_h) \leqslant a(u, u - v_h) + a(u, v - u_h - u + v_h) - f(v - u).$$

On the other hand, we have

$$-a(u_h, u - u_h) \leqslant -a(u_h, u - v_h) - f(v_h - u_h).$$

By adding these two inequalities, we obtain

$$\begin{aligned}a(u - u_h, u - u_h) &\leqslant a(u - u_h, u - v_h) + a(u, v - u_h - u + v_h) - f(v - u_h - u + v_h), \\ &\leqslant a(u - u_h, u - v_h) + a(u, v_h - u) - f(v_h - u) + a(u, v - u_h) - f(v - u_h).\end{aligned}$$

The continuity of a on $V_h + V$ and the Young inequality give, for $\delta > 0$,

$$\begin{aligned}a(u - u_h, u - u_h) \leqslant {}& C(\delta \|u - u_h\|_h^2 + \delta^{-1} \|u - v_h\|_h^2) + a(u, v_h - u) \\ &- f(v_h - u) + a(u, v - u_h) - f(v - u_h).\end{aligned} \tag{16}$$

The two remaining terms in (15) are treated by using the continuity of a;

$$a(v_h - u, v_h - u_h) \leqslant C(\delta^{-1}\|v_h - u\|_h^2 + \delta\,\|v_h - u_h\|_h^2),$$
$$a(u - u_h, v_h - u) \leqslant C(\delta^{-1}\|v_h - u\|_h^2 + \delta\,\|u - u_h\|_h^2). \tag{17}$$

Now, we get the result for δ sufficiently small adding estimates (14)–(17), and thanks to the triangle inequality

$$\|u - u_h\|_h \leqslant \|u - v_h\|_h + \|v_h - u_h\|_h.$$

The above error estimate allows us to prove the following result. □

Theorem 3. *Under hypothesis* (10), ($H0^{nc}$) *and if the family* $(K_h)_{h>0}$ *approaches* K (*Definition* 1), *we have*

$$\lim_{h\to 0} \|u_h - u\|_h = 0,$$

where u *and* u_h *are the solutions to* (11) *and* (12), *respectively*.

Proof. First, by a standard method it is possible to prove that u_h is bounded. Then, from ($H0^{nc}$) there exists a subsequence, still denoted by u_h, which converges weakly to an element $u \in V$. Since K_h approaches K, we necessarily have $u \in K$. We now check that u is a solution to (11). Let $v \in K$; there exists $\tilde{v}_h \in K_h$ such that

$$\lim_{h\to 0} \|\tilde{v}_h - v\|_h = 0.$$

From (12), we have

$$a(u_h, \tilde{v}_h - u_h) \geqslant f(\tilde{v}_h - u_h).$$

The passage to the limit implies

$$a(u, v - u) \geqslant f(v - u).$$

We deduce from the uniqueness of the limit that the whole sequence converges weakly to u. Using estimate (13) with $v = u$ and $v_h = \tilde{u}_h$ where $\tilde{u}_h$ satisfies

$$\lim_{h\to 0} \|\tilde{u}_h - u\|_h = 0$$

and taking into account the weak convergence of u_h to u, we conclude

$$\lim_{h\to 0} \|u_h - u\|_h = 0. \qquad \square$$

Remark 3. In the conforming case the proof can be found in [14,16,17].

Applying the preceding theorem to our initial problem, we get the following results.

Proposition 2. 1. *Assuming that the family* $(K_h)_{h>0}$ *approaches* K *in the sense of Definition* 1 *and for* $\varepsilon > 0$ *fixed we have*:

$$\lim_{h\to 0} \|u_h^\varepsilon - u^\varepsilon\|_h = 0,$$

where u^ε *and* u_h^ε *are the solutions to* (1) *and* (6), *respectively*.

2. *Assuming that the family* $(K_{0h})_{h>0}$ *approaches* K_0 *in the sense of Definition* 1, *we have*

$$\lim_{h\to 0} \|u_h^0 - u^0\|_h = 0,$$

where u^0 *and* u_h^0 *are the solutions to* (2) *and* (7), *respectively*.

3.3. Uniform convergence

We are now interested in studying the uniform convergence with respect to ε of the Galerkin's method. We will proceed through several steps by separating the study in the neighborhood of $\varepsilon = 0$ and the study on the interval $[\varepsilon_0, 1]$ with $\varepsilon_0 > 0$.

Proposition 3. *Let* u^ε *and* u_h^ε *be the solutions to* (1) *and* (6), *respectively. If* $(K_h)_{h>0}$ *and* $(K_{0h})_{h>0}$ *approach* K *and* K_0, *respectively in the sense of Definition 1*, we have

$$\lim_{\varepsilon, h\to 0} \|u_h^\varepsilon - u^\varepsilon\|_h = 0.$$

Proof. The triangular inequality gives

$$\|u^\varepsilon - u_h^\varepsilon\|_h \leqslant \|u^\varepsilon - u^0\|_h + \|u^0 - u_h^0\|_h + \|u_h^0 - u_h^\varepsilon\|_h. \tag{18}$$

Now, according to Theorem 1 and Proposition 2, we have, respectively,

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - u^0\|_h = 0$$

and

$$\lim_{h\to 0} \|u^0 - u_h^0\|_h = 0.$$

In order to show that

$$\lim_{\varepsilon, h\to 0} \|u_h^0 - u_h^\varepsilon\|_h = 0 \tag{19}$$

we proceed as follows. First of all, let us prove that the sequence $(u_h^\varepsilon)_{\varepsilon,h}$ is uniformly bounded in ε and h. The fact that K_{0h} approaches K_0 ensures the existence of a bounded sequence $(v_h^0)_h$ in K_{0h}. From (6) we have

$$a^\varepsilon(u_h^\varepsilon, v_h^0 - u_h^\varepsilon) \geqslant L(v_h^0 - u_h^\varepsilon).$$

Proceeding as in the proof of Theorem 1, we deduce that

$$\|u_h^\varepsilon\|_h \leqslant C, \quad a_0(u_h^\varepsilon, u_h^\varepsilon) \leqslant C_\varepsilon,$$

where C is a positive constant independent of ε and h. This allows us to deduce the existence of a subsequence, still denoted by $(u_h^\varepsilon)_{\varepsilon,h}$, weakly convergent to an element $w^0 \in K_0$ when $\varepsilon, h \to 0$.

Since K_{0h} approaches K_0, it follows that for all $w \in K_0$ there exists $w_h \in K_{0h}$ which converges strongly in V to w. Inequality (6) gives

$$a_1(u_h^\varepsilon, w_h - u_h^\varepsilon) + \varepsilon^{-1} a_0(u_h^\varepsilon, -u_h^\varepsilon) \geqslant L(w_h - u_h^\varepsilon),$$

hence

$$a_1(u_h^\varepsilon, w_h - u_h^\varepsilon) \geqslant L(w_h - u_h^\varepsilon).$$

By passing to the limit when $\varepsilon, h \to 0$ and using the lower semi-continuity of a_1, we obtain

$$a_1(w^0, w - w^0) \geqslant L(w - w^0).$$

The uniqueness of the solution to Problem (2) implies $w^0 = u^0$.

Thanks to the coercivity of a^ε, it follows:

$$\begin{aligned}\alpha \|u_h^0 - u_h^\varepsilon\|_h^2 &\leqslant a^\varepsilon(u_h^\varepsilon - u_h^0, u_h^\varepsilon - u_h^0)\\ &= a^\varepsilon(u_h^\varepsilon, u_h^\varepsilon - u_h^0) - a_1(u_h^0, u_h^\varepsilon - u_h^0)\\ &\leqslant L(u_h^\varepsilon - u_h^0) - a_1(u_h^0, u_h^\varepsilon - u_h^0).\end{aligned}$$

Passing to the limit and taking into account the weak convergence of u_h^ε to u^0 and the strong convergence of u_h^0 to u^0, we obtain (19). Gathering these three properties, the proof of the proposition is completed. □

Proposition 4. *If $(K_h)_{h>0}$ approaches K, then we have*

$$\forall \varepsilon_0 \in]0,1], \quad \lim_{h\to 0} \sup_{\varepsilon_0 \leqslant \varepsilon \leqslant 1} \|u_h^\varepsilon - u^\varepsilon\|_h = 0,$$

where u^ε and u_h^ε are the solutions to (1) and (6), respectively.

Proof. We adopt a method used in [9]. For any $h > 0$, we consider the function

$$g_h(\varepsilon) = \|u^\varepsilon - u_h^\varepsilon\|_h.$$

Since the form a^ε does not depend in a singular manner of ε, the classical properties of the variational inequalities give that the functions $\varepsilon \to u^\varepsilon$ and $\varepsilon \to u_h^\varepsilon$ are continuous on $[\varepsilon_0, 1]$. It follows that $g_h(\varepsilon)$ is continuous on $[\varepsilon_0, 1]$. Hence, the Weierstrass theorem gives

$$\exists \varepsilon_h \text{ such that : } \sup_{\varepsilon_0 \leqslant \varepsilon \leqslant 1} \|u^\varepsilon - u_h^\varepsilon\|_h = \|u^{\varepsilon_h} - u_h^{\varepsilon_h}\|_h.$$

Since the sequence $(\varepsilon_h)_h$, is bounded, we can extract a subsequence, still denoted by $(\varepsilon_h)_h$, which converges to $\bar{\varepsilon}$.

The preceding estimates and techniques permit us to show that $(u_h^{\varepsilon_h})_h$ is bounded. So, there exists a subsequence, still denoted $(u_h^{\varepsilon_h})_h$, which converges weakly to an element $w \in K$. We are going to prove that $w = u^{\bar{\varepsilon}}$ and that $(u_h^{\varepsilon_h})_h$ converges strongly to $u^{\bar{\varepsilon}}$. The triangle inequality gives

$$\|u^{\bar{\varepsilon}} - u_h^{\varepsilon_h}\|_h \leqslant \|u^{\bar{\varepsilon}} - u_h^{\bar{\varepsilon}}\|_h + \|u_h^{\bar{\varepsilon}} - u_h^{\varepsilon_h}\|_h. \tag{20}$$

Since K_h approaches K, Proposition 2 implies

$$\lim_{h\to 0} \|u^{\bar\varepsilon} - u_h^{\bar\varepsilon}\|_h = 0.$$

We have to prove that the second term of the right-hand side of (20) also tends to 0.

The uniform coercivity of a^{ε_h} gives

$$\alpha\|u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}\|_h^2 \leqslant a^{\varepsilon_h}(u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}).$$

Since $u_h^{\varepsilon_h}$ is the solution to (6) for $\varepsilon = \varepsilon_h$ and $u_h^{\bar\varepsilon} \in K_h$, it follows:

$$a^{\varepsilon_h}(-u_h^{\varepsilon_h}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}) \leqslant -L(u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}).$$

Using the definition of a^ε, we can write

$$a^{\varepsilon_h}(u_h^{\bar\varepsilon}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}) = a^{\bar\varepsilon}(u_h^{\bar\varepsilon}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}) + (\varepsilon_h^{-1} - \bar\varepsilon^{-1})a_0(u_h^{\bar\varepsilon}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}).$$

Adding these two last relations, we get

$$a^{\varepsilon_h}(u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}) \leqslant (a^{\bar\varepsilon}(u_h^{\bar\varepsilon}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}) - L(u_h^{\bar\varepsilon} - u_h^{\varepsilon_h})) + (\varepsilon_h^{-1} - \bar\varepsilon^{-1})a_0(u_h^{\bar\varepsilon}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}).$$

Using

$$a^{\bar\varepsilon}(u_h^{\bar\varepsilon}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}) - L(u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}) \leqslant 0$$

we obtain

$$\alpha\|u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}\|_h^2 \leqslant (\varepsilon_h^{-1} - \bar\varepsilon^{-1})a^0(u_h^{\bar\varepsilon}, u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}).$$

Since $u_h^{\bar\varepsilon}$ converges strongly and $u_h^{\varepsilon_h}$ converges weakly, it follows:

$$\lim_{h\to 0} \|u_h^{\bar\varepsilon} - u_h^{\varepsilon_h}\|_h = 0.$$

From (20), we deduce that

$$\lim_{h\to 0} \|u^{\bar\varepsilon} - u_h^{\varepsilon_h}\|_h = 0.$$

The uniqueness of $u^{\bar\varepsilon}$ implies the convergence of the whole sequence.

Gathering the previous results, we obtain

$$\begin{aligned}\lim_{h\to 0} \sup_{\varepsilon_0 \leqslant \varepsilon \leqslant 1} \|u_h^\varepsilon - u^\varepsilon\|_h &= \lim_{h\to 0} \|u^{\varepsilon_h} - u_h^{\varepsilon_h}\|_h \\ &\leqslant \lim_{h\to 0} \|u^{\varepsilon_h} - u^{\bar\varepsilon}\|_h + \lim_{h\to 0} \|u^{\bar\varepsilon} - u_h^{\varepsilon_h}\|_h = 0,\end{aligned}$$

and the proof is completed. □

Now, with Propositions 3 and 4, it is straightforward to get the following theorem.

Theorem 4. *Let u^ε and u_h^ε be the solutions to* (1) *and* (6), *respectively. If $(K_h)_{h>0}$ approaches K, and if $(K_{0h})_{h>0}$ approaches K_0, then*

$$\lim_{h\to 0} \sup_{0<\varepsilon \leqslant 1} \|u_h^\varepsilon - u^\varepsilon\|_h = 0.$$

We conclude that if the conditions of Theorem 4 are satisfied, we will have a method without numerical locking. In practice, these conditions are not always satisfied for the conforming finite element methods, (cf. [25]). It is easier to satisfy them in the nonconforming case $V_h \not\subset V$, which we will examine now.

Remark 4. We can prove also the following result, as in the linear case [9],

$$\lim_{h\to 0} \sup_{0<\varepsilon\leqslant 1} |u_h^\varepsilon - u^\varepsilon|_{a^\varepsilon} = 0,$$

where $|\cdot|_{a^\varepsilon}$ is the semi-norm given by $|v|_{a^\varepsilon} = (a_\varepsilon(v,v))^{1/2}$.

4. Application

The transmission problems with Signorini boundary conditions are commonly used in several problems of structural mechanics, heat transfer, etc. In this paper, we present the mechanical model of unilateral contact in linear anti-plane elasticity. This will highlight some technical difficulties which appear when we apply the uniform convergence theorem given in the preceding section.

Let us start with a brief description of a unilateral contact problem involving an elastic body and a thin shell in general situation. Let Ω^+ be an elastic body (an open set) whose boundary is denoted by $\partial\Omega^+$. Let us suppose that $\partial\Omega^+ = \Sigma \cup \Gamma^+$ where both Σ and Γ^+ are of nonvanishing measure. This elastic body has a thin shell, noted Ω_ε^-, grafted onto Σ. The parameter ε characterizes the thickness of the thin shell, and it will tend to 0. The boundary Γ^+ is clamped. Both bodies behave according to linear and isotropic law which we suppose to be characterized by their respective Lamé coefficients λ, μ and $\lambda/\varepsilon, \mu/\varepsilon$. The thin shell behaves as a stiffener on Σ. During the deformation, the system can be in contact on a part, denoted $\Gamma_{C,\varepsilon}$, of its boundary with a rigid obstacle.

For the sake of simplicity, we suppose that Σ is straight. The following geometry is a typical case of a straight interface (see Fig. 1):

$$\Omega^+ =]0,1[\times]0,1[, \qquad \Omega_\varepsilon^- =]0,1[\times]-\varepsilon,0[.$$

The problem of stiff transmission with Signorini boundary conditions is given by

$$\begin{aligned}
&-\Delta u^\varepsilon = f \quad \text{in } \Omega^+,\\
&-\frac{1}{\varepsilon}\Delta u^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon^-,\\
&u^\varepsilon = 0 \quad \text{on } \Gamma_{D,\varepsilon} = \Gamma^+ \cup \Gamma_{D,\varepsilon}^-,\\
&u^{+,\varepsilon} = u^{-,\varepsilon} \quad \text{on } \Sigma,\\
&\partial_n u^{+,\varepsilon} = \frac{1}{\varepsilon}\partial_n u^{-,\varepsilon} \quad \text{on } \Sigma,\\
&u^\varepsilon \geqslant 0, \quad \partial_n u^\varepsilon \geqslant 0, \quad u^\varepsilon \partial_n u^\varepsilon = 0 \quad \text{on } \Gamma_{C,\varepsilon},
\end{aligned} \tag{21}$$

where n denotes the unit outward normal to Ω_ε^- on $\Gamma_{C,\varepsilon}$ and the unit outward normal to Ω^+ on Σ. The superscripts $+$ and $-$ in the above transmission conditions indicate that the trace on Σ

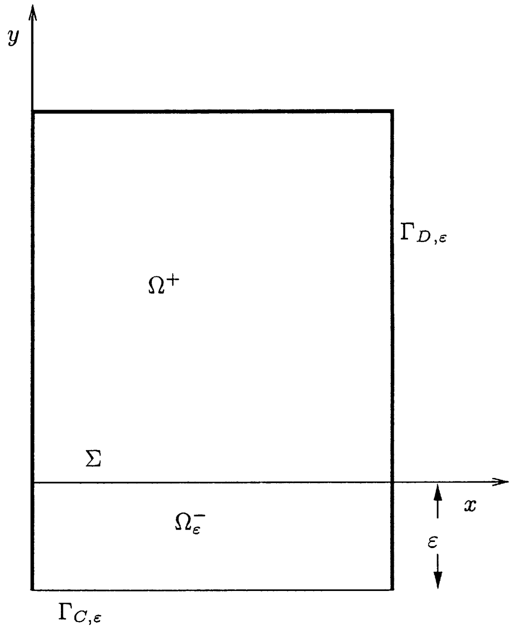

Fig. 1. Domain with straight interface.

corresponds to the values of u^ε in Ω^+ and Ω_ε^-, respectively. Let V_ε be the space

$$V_\varepsilon := \{v \in H^1(\Omega_\varepsilon); \quad v = 0 \text{ on } \Gamma_{D,\varepsilon}\}$$

and K_ε the convex subset defined by

$$K_\varepsilon := \{v \in V_\varepsilon; \quad v \geqslant 0 \text{ on } \Gamma_{C,\varepsilon}\}.$$

The variational formulation of (21) can be written as follows:

$$u^\varepsilon \in K_\varepsilon,\ \forall v \in K_\varepsilon, \quad \int_{\Omega_\varepsilon} \mu_\varepsilon \nabla u^\varepsilon \cdot \nabla(v - u^\varepsilon)\,\mathrm{d}\Omega_\varepsilon \geqslant \int_{\Omega_\varepsilon} f(v - u^\varepsilon)\,\mathrm{d}\Omega_\varepsilon. \tag{22}$$

This problem depends singularly on ε when ε goes to 0. Actually, two difficulties appear. First, the thickness of the thin shell tends to 0, and secondly, the stiffness coefficients tend to infinity.

Using the scale change $\tilde{y} = y/\varepsilon$, $\tilde{v}(x, \tilde{y}) = v(x, y), -\varepsilon < y < 0$, the thin domain Ω_ε^- becomes $\Omega^- =]0, 1[\times] - 1, 0[$, the space V_ε turns into

$$V := \{(v^+, v^-) \in H^1(\Omega^+) \times H^1(\Omega^-);\ v^+ = v^- \text{ on } \Sigma,\ v = 0 \text{ on } \Gamma_D\}$$

and the convex subset K_ε turns into

$$K := \{v \in V; \quad v^- \geqslant 0 \text{ on } \Gamma_C\},$$

where Ω is the interior of $\bar{\Omega}^- \cup \bar{\Omega}^-$. The remaining notations can be found in Fig. 2. The space V is equipped with the usual norm of $H^1(\Omega) : \|.\|_{1,\Omega}$. To simplify the notations, $\tilde{v}$ will be denoted by v, and the solution $\tilde{u}^\varepsilon$ on the new domain Ω will simply be denoted by u^ε. Problem (22) can thus

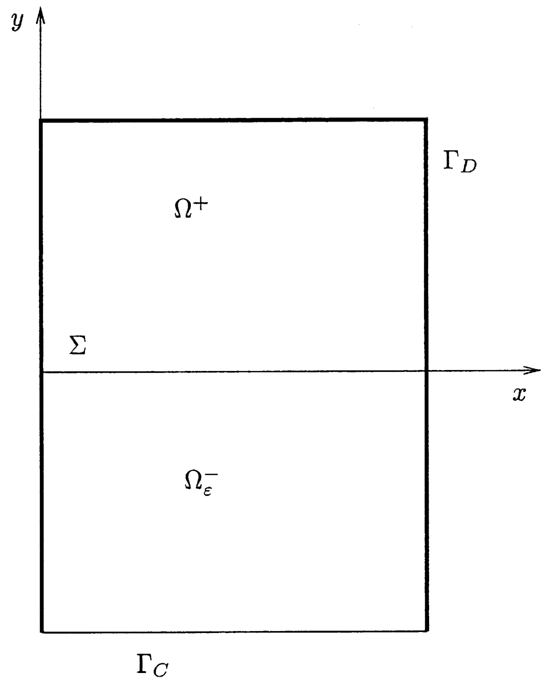

Fig. 2. Domain after change of variable.

be written under the following form:

$$u^\varepsilon \in K, \ \forall v \in K,$$
$$a^+(u^{+,\varepsilon}, v^+ - u^{+,\varepsilon}) + a_x^-(u^{-,\varepsilon}, v^- - u^{-,\varepsilon}) + \varepsilon^{-2} a_y^-(u^{-,\varepsilon}, v^- - u^{-,\varepsilon}) \geqslant (f, v^+ - u^{+,\varepsilon}), \quad (23)$$

where

$$a^+(u^+, v^+) = \int_{\Omega^+} \nabla u^+ \cdot \nabla v^+ \, \mathrm{d}\Omega^+, \quad a_x^-(u^-, v^-) = \int_{\Omega^-} \partial_x u^- \partial_x v^- \, \mathrm{d}\Omega^-,$$
$$a_y^-(u^-, v^-) = \int_{\Omega^-} \partial_y u^- \partial_y v^- \, \mathrm{d}\Omega^-, \quad (f, v^+) = \int_{\Omega^+} f \, v^+ \, \mathrm{d}\Omega^+.$$

Problem (23) has a simple and explicit dependence on ε and it fits into the general framework studied in the previous sections.

Proposition 5. *The sequence $(u^\varepsilon)_\varepsilon$ converges strongly in V to u^0, the solution to the following problem*:

$$\begin{cases} u^0 \in K_0, \ \forall v \in K_0, \\ a^+(u^{+,0}, v^+ - u^{+,0}) + a_x^-(u^{-,0}, v^- - u^{-,0}) \geqslant (f, v^+ - u^{+,0}). \end{cases} \quad (24)$$

Remark 5. Problem (24) is equivalent to the Ventcel–Signorini problem (cf. [19,25])

$$
\begin{aligned}
&-\Delta u^{+,0} = f \quad \text{in } \Omega^+,\\
&u^{+,0} = 0 \quad \text{on } \Gamma^+,\\
&u^{+,0} \geqslant 0,\ (\partial_n u^{+,0} - \partial^2_{xx} u^{+,0}) \geqslant 0 \quad \text{on } \Sigma,\\
&u^{+,0}(\partial_n u^{+,0} - \partial^2_{xx} u^{+,0}) = 0 \quad \text{on } \Sigma.
\end{aligned} \tag{25}
$$

4.1. Conforming finite elements method

Let $(\mathscr{T}_h)_{h>0}$ be a regular family of triangulation (cf. [11]) of the domain $\bar{\Omega}$ into triangles of diameter no greater than h, compatible with the decomposition of Ω in Ω^+ and Ω^-

$$\bar{\Omega} = \bigcup_{T \in \mathscr{T}_h} T.$$

We introduce the finite-dimensional subspace

$$V_h := \{v_h \in V;\ \forall T \in \mathscr{T}_h, \quad v|_T \in \mathbb{P}_1(T)\},$$

where for each $T \in \mathscr{T}_h$, the space of polynomials of degree at most 1 on T is denoted by $\mathbb{P}_1(T)$. Let K_h be the discrete convex subset

$$K_h = \{v_h \in V_h; \quad v_h \geqslant 0 \text{ on } \Gamma_C\}.$$

We remark that $K_h \subseteq K$. The discrete version of Problem (23) can be written as

$$
\begin{aligned}
&u_h^\varepsilon \in K_h,\ \forall v_h \in K_h,\\
&\quad a^+(u_h^{+,\varepsilon}, v_h^+ - u_h^{+,\varepsilon}) + a_x^-(u_h^{+,\varepsilon}, v_h^+ - u_h^{+,\varepsilon}) + \varepsilon^{-2} a_y^-(u_h^{-,\varepsilon}, v^- - u_h^{-,\varepsilon}) \geqslant (f, v_h^- - u_h^{-,\varepsilon}).
\end{aligned} \tag{26}
$$

From Proposition 2, the sequence u_h^ε converges to u_h^0 when ε tends to 0, the solution to the following problem

$$u_h^0 \in K_{0h},\ \forall v_h \in K_{0h}, \quad a^+(u_h^{+,0}, v_h^+ - u_h^{+,0}) + a_x^-(u_h^{-,0}, v_h^- - u_h^{-,0}) \geqslant (f, v_h^+ - u_h^{+,0}), \tag{27}$$

where

$$K_{0h} = \{v_h \in V_h \colon \partial_y v_h^- = 0 \quad \text{in } \Omega^- \text{ and } v_h^-|_{\Gamma_C} \geqslant 0\}.$$

We note that the discretization is conformal since

$$K_{0h} \subseteq K_0.$$

Using the Falk lemma [13], it is possible to prove that if $u^\varepsilon \in H^2(\Omega^+ \cup \Omega^-)$ the following estimate holds:

$$\exists C > 0 \colon \|u^\varepsilon - u_h^\varepsilon\| \leqslant C\varepsilon^{-2} h^{3/4}(\|u^{+,\varepsilon}\|_{2,\Omega^+} + \|u^{-,\varepsilon}\|_{2,\Omega^-}). \tag{28}$$

This estimate highlights the fact that one needs to choose $h \ll \varepsilon$.

As we saw previously, having a uniform convergence requires conditions on the discrete convexes.

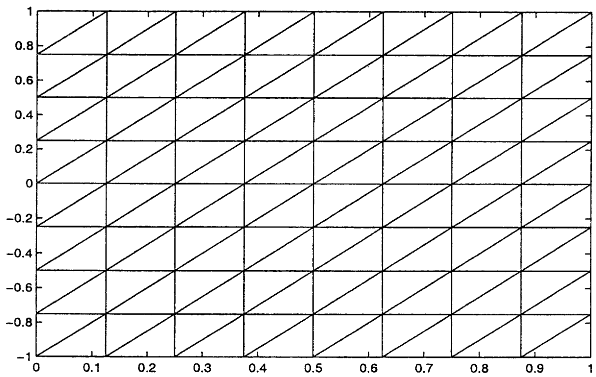

Fig. 3. Structured mesh of the domain $\bar{\Omega}$.

We now consider a regular structured mesh of $\bar{\Omega}$ shown in Fig. 3. We obtain the following result generalizing the linear case [23].

Theorem 5. *In the case of meshes of the type as shown in Fig.* 3 (*i.e. structured meshes*), *uniform convergence takes place*

$$\lim_{h\to 0} \sup_{0<\varepsilon\leqslant 1} \|u^\varepsilon - u_h^\varepsilon\|_{1,\Omega} = 0.$$

The proof of this theorem relies on the following lemma.

Lemma 1. *We have*

$$\overline{\mathscr{D}(\bar{\Omega}) \cap K_0} = K_0,$$

where $\overline{\mathscr{D}(\bar{\Omega}) \cap K_0}$ *is the adherence of* $\mathscr{D}(\bar{\Omega}) \cap K_0$ *in* $H^1(\Omega)$.

Proof. First of all, we remark that K_0 can be identified to the convex subset K^+

$$K^+ = \{v \in H^1(\Omega^+);\ v|_\Sigma \in H_0^1(\Sigma),\ v|_{\Gamma^+} = 0,\ v \geqslant 0 \text{ on } \Sigma\},$$

where the space X is defined as

$$X := \{v \in H^1(\Omega^+);\ v|_\Sigma \in H_0^1(\Sigma),\ v|_{\Gamma^+} = 0\},$$

and it is equipped with the norm $(\|.\|_{1,\Omega^+}^2 + \|.\|_{1,\Sigma}^2)^{1/2}$. So, it is enough to prove that

$$\overline{\mathscr{D}(\overline{\Omega^+}) \cap K^+} = K^+,$$

where $\overline{\mathscr{D}(\overline{\Omega^+}) \cap K^+}$ is the adherence of $\mathscr{D}(\overline{\Omega^+}) \cap K^+$ in X.

Let $v \in K^+$. Since $v|_\Sigma \in H_0^1(\Sigma)$ and $v|_\Sigma \geqslant 0$; following [14], there exists a sequence $\tilde{\phi}_n \in \mathscr{D}(\Sigma)$ such that

$$\tilde{\phi}_n \geqslant 0 \text{ on } \Sigma, \quad \tilde{\phi}_n \to v|_\Sigma \text{ in } H_0^1(\Sigma).$$

Thanks to the extension theorem, we can construct a sequence $\phi_n \in \mathscr{D}(\bar{\Omega}^+)$ such that

$$\phi_{n|_{\Gamma^+}} = 0 \quad \text{and} \quad \phi_n|_\Sigma = \tilde{\phi}_n.$$

The continuity of the extension operator ensures the existence of an element $\phi \in H^{3/2}(\Omega^+)$ such that

$$\phi_n \to \phi \quad \text{in } H^{3/2}(\Omega^+).$$

Therefore, we have

$$\phi|_{\Gamma^+} = 0 \quad \text{and} \quad \phi|_\Sigma = v|_\Sigma.$$

We now choose

$$w = v - \phi.$$

We remark that $w \in H_0^1(\Omega^+)$. The density of $\mathscr{D}(\Omega^+)$ in $H_0^1(\Omega^+)$ implies the existence of a sequence $w_n \in \mathscr{D}(\Omega^+)$ such that

$$w_n \to w \quad \text{in } H^1(\Omega^+).$$

The sequence $v_n = w_n + \phi_n$ satisfies

$$v_n \in \mathscr{D}(\overline{\Omega^+}) \cap K^+ \quad \text{and} \quad v_n \to v \quad \text{in } X,$$

which ends the proof of the lemma. □

Proof of the theorem. It is enough to verify that K and K_0 are correctly approached by K_h and K_{0h}, respectively. It is well known that K_h approaches K (see. [14,15]). The uniformity of the mesh permits us to deduce that $I_h v \in K_{0h}$, for all regular $v \in K_0$, where I_h is the Lagrange interpolation operator of degree 1. The usual interpolation error estimates give

$$\lim_{h \to 0} \|v - I_h v\|_{1,\Omega} = 0.$$

Thanks to the density of $\mathscr{D}(\bar{\Omega}) \cap K_0$ in K_0, we conclude that K_{0h} approaches K_0. □

Unilateral contact problems generally do not admit analytical solution. Since we have established a convergence result, we will choose as the reference solution, denoted $u^\varepsilon_{\text{ref}}$, the solution computed on uniform very refined mesh. The computation is done with the standard finite element method combined with an Uzawa algorithm. It is obvious from Fig. 4, that the reference solution has a particular geometric behavior: its value along the normal to Σ is constant and nonnegative over Ω^-. This confirms the result obtained previously that $\partial_y u^\varepsilon \to 0$ in Ω^- when $\varepsilon \to 0$.

We now compute the solution over the structured mesh shown in Fig. 3. Fig. 5 gives the discrete solution. We see that this solution is in agreement with the reference solution.

In what follows, we use an arbitrary triangulation; for example, the mesh given in Fig. 6. The discrete solution computed with this mesh (shown in Fig. 7) vanishes on Ω^-. Hence, the exact solution is not well approached. It is a clear manifestation of the locking phenomenon. We remark that this discrete solution does not show any numerical instability: it does not blow up or oscillate quickly, and its restriction to Ω^+ is the solution to a standard Laplace problem with an homogeneous Dirichlet boundary condition, without any connection to Problem (25). We note that in some cases the locking phenomenon is represented by a margin between the computed solution and the exact solution (see [23]).

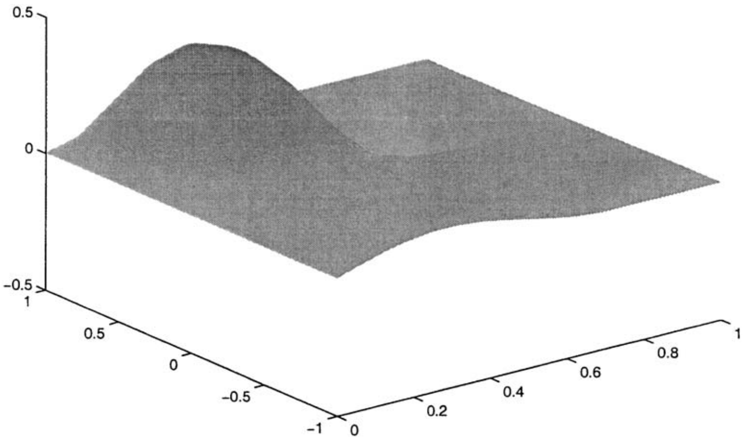

Fig. 4. Reference solution; $\varepsilon = 10^{-6}$.

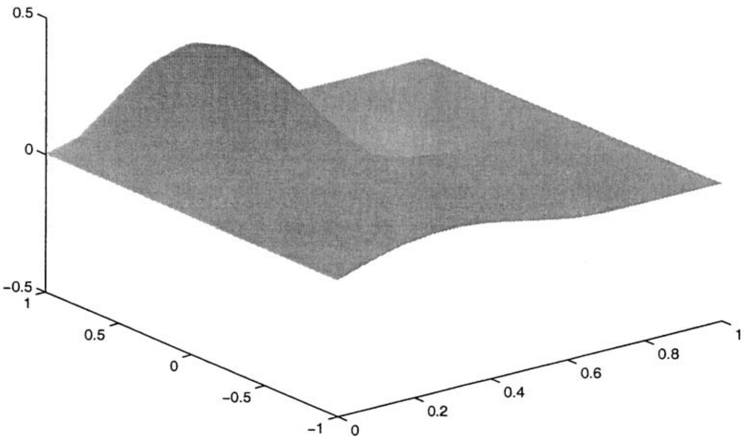

Fig. 5. Approximated solution; $\varepsilon = 10^{-6}$.

4.2. Nonconforming finite element approximation

In this section, we are interested in studying the approximation of Problem (23) by $\mathbb{P}_1$ nonconforming finite element method which is robust in the linear case [8]. Our aim is to show that this method is also robust in the nonlinear case.

The nonconforming finite-dimensional subspace is given by

$$
\begin{aligned}
V_h^{\rm nc} := \{ & v_h \in L^2(\Omega);\ \forall T \in \mathscr{T}_h, \quad v_h|_T \in \mathbb{P}_1(T),\\
& v_h \text{ continuous at the midpoints of each internal edges},\\
& v_h \text{ vanishes on the midpoints setting on } \Gamma_D \}.
\end{aligned}
$$

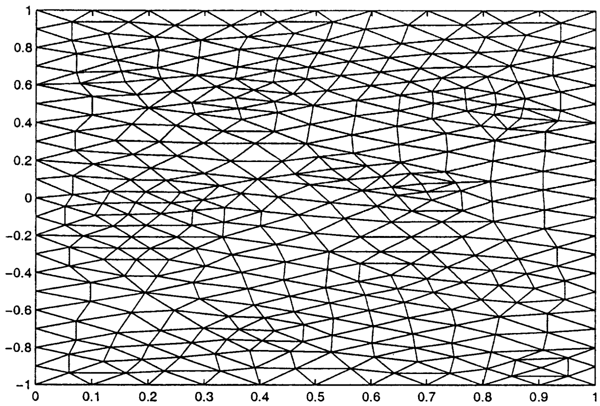

Fig. 6. Unstructured mesh.

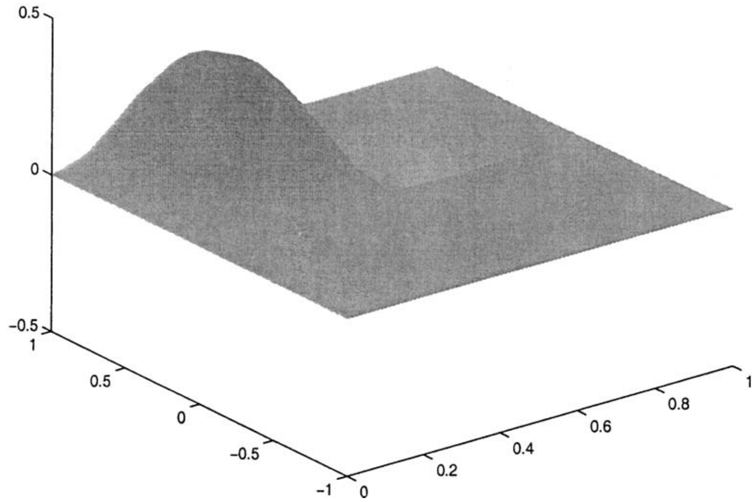

Fig. 7. Discrete solution; $\varepsilon = 10^{-6}$.

The space V_h^{nc} is equipped with the broken norm of $H^1(\Omega)$

$$\|v\|_{1,\Omega_h} = \left(\sum_{T \in \mathscr{T}_h} \|v\|_{1,T}^2 \right)^{1/2}.$$

The nonconforming interpolation operator is given by (see [12])

$$I_h^{\mathrm{nc}} : V \to V_h^{\mathrm{nc}}$$

$$\forall v \in V, \ \ (I_h v)(m_{T'}) = \frac{1}{|T'|} \int_{T'} v \, \mathrm{d}\Gamma,$$

for each T' of the triangulation $\mathcal{T}_h$ of the middle $m_{T'}$, where $|T'|$ is the length of T'. We have the following estimate:

$$\forall v \in H^2(\Omega^+ \cup \Omega^-), \quad \|v - I_h^{\mathrm{nc}} v\|_{1,\Omega_h} \leqslant Ch(|v|_{2,\Omega^+} + |v|_{2,\Omega^-}).$$

We can extend the definition of a^+, a_x^- and a_y^- on $(V + V_h^{\mathrm{nc}}) \times (V + V_h^{\mathrm{nc}})$ by putting

$$a^+(u,v) = \sum_{T \in \mathcal{T}_h^+} \int_T \nabla u . \nabla v \, \mathrm{d}T,$$

$$a_x^-(u,v) = \sum_{T \in \mathcal{T}_h^-} \int_T \partial_x u \partial_x v \, \mathrm{d}T,$$

$$a_y^-(u,v) = \sum_{T \in \mathcal{T}_h^-} \int_T \partial_y u \partial_y v \, \mathrm{d}T,$$

where $\mathcal{T}_h^+$, $\mathcal{T}_h^-$ is the restriction of $\mathcal{T}_h$ on Ω^+ and Ω^-, respectively. These forms are obviously continuous on $(V + V_h^{\mathrm{nc}}) \times (V + V_h^{\mathrm{nc}})$ with regard to the norm $\|.\|_{1,\Omega_h}$ and with continuity constants independent of h. Furthermore, we have the following properties:

- the bilinear forms are symmetric on $(V + V_h^{\mathrm{nc}}) \times (V + V_h^{\mathrm{nc}})$;
- for all $\varepsilon > 0$, the bilinear form given by

  $$a^{\mathrm{nc},\varepsilon}(.,.) = a^+(.,.) + a_x^-(.,.) + \varepsilon^{-2} a_y^-(.,.)$$

  is uniformly elliptic (in ε and h) on V_h^{nc} (see [9]).

The kernel of $a_y^-(.,.)$ on V_h^{nc} is given by

$$G_h^{\mathrm{nc}} = \{v_h \in V_h^{\mathrm{nc}};\ \forall T \in \mathcal{T}_h^-,\ \partial_y v_h|_T = 0\}.$$

The discrete formulation with $\mathbb{P}_1$ nonconforming method can be written as

$$u_h^{\mathrm{nc},\varepsilon} \in K_h^{\mathrm{nc}},\ \forall v_h \in K_h^{\mathrm{nc}}, \quad a^+(u_h^{\mathrm{nc},\varepsilon}, v_h - u_h^{\mathrm{nc},\varepsilon}) + a_x^-(u_h^{\mathrm{nc},\varepsilon}, v_h - u_h^{\mathrm{nc},\varepsilon}) + \varepsilon^{-2} a_y^-(u_h^{\mathrm{nc},\varepsilon}, v - u_h^{\mathrm{nc},\varepsilon})$$
$$\geqslant (f, v_h - u_h^{\mathrm{nc},\varepsilon}), \tag{29}$$

as well as for the Signorini–Ventcel problem

$$u_h^{\mathrm{nc},0} \in K_{0h}^{\mathrm{nc}},\ \forall v_h \in K_{0h}^{\mathrm{nc}}, \quad a^+(u_h^{\mathrm{nc},0}, v_h - u_h^{\mathrm{nc},0}) + a_x^-(u_h^{\mathrm{nc},0}, v_h - u_h^{\mathrm{nc},0}) \geqslant (f, v_h - u_h^{\mathrm{nc},0}), \tag{30}$$

where $K_{0h}^{\mathrm{nc}} = K_h^{\mathrm{nc}} \cap G_h^{\mathrm{nc}}$. We can show the following result (see, [25]).

Theorem 6. *If the solution u^ε to (23) is in $H^2(\Omega^+ \cup \Omega^-)$, we have the estimate*

$$\|u^\varepsilon - u_h^{\varepsilon,\mathrm{nc}}\|_{1,\Omega_h} \leqslant C\varepsilon^{-2} h^{3/4} (\|u^\varepsilon\|_{2,\Omega^-} + \|u^\varepsilon\|_{2,\Omega^-}). \tag{31}$$

We remark, as in $\mathbb{P}_1$ conforming approximation method, that the error blow up as soon as ε is smaller than h.

In order to apply Theorem 4, we give the following lemma.

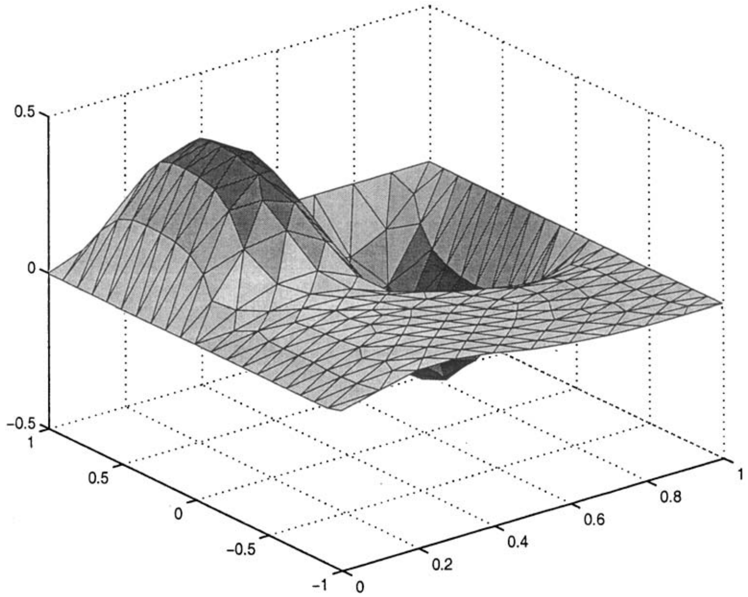

Fig. 8. Approximate solution by nonconforming $\mathbb{P}_1$; $\varepsilon = 10^{-6}$.

Lemma 2. *The discrete convex subset* K_h^{nc} (*resp.* K_{0h}^{nc}) *approaches* K (*resp.* K_0).

Proof. Let $v \in K$. Using the definition of the nonconforming interpolation of v, given by

$$(I_h^{\mathrm{nc}} v)(m_{T'}) = \frac{1}{|T'|} \int_{T'} v \, \mathrm{d}\Gamma,$$

we deduce that if $T' \subset \Gamma_C$ we have $(I_h^{\mathrm{nc}} v)(m_{T'}) \geqslant 0$ which implies that $I_h^{\mathrm{nc}} v \in K_h^{\mathrm{nc}}$. Taking into the account the estimate

$$\forall v \in H^2(\Omega^+ \cup \Omega^-), \quad \|v - I_h^{\mathrm{nc}} v\|_{1,\Omega_h} \leqslant Ch(|v|_{2,\Omega^+} + |v|_{2,\Omega^-}), \tag{32}$$

and the fact that $\mathscr{D}(\bar{\Omega}) \cap K$ is dense in K, we can conclude that K_h^{nc} approaches K.

The second approximation result can be obtained in a similar way. If $v \in K_0$, we saw that $I_h^{\mathrm{nc}} v \in K_h^{\mathrm{nc}}$. Using the Green formula we can easily show that $I_h^{\mathrm{nc}} v \in G_h^{\mathrm{nc}}$ (see [9]), where $I_h^{\mathrm{nc}} v \in K_{0h}$. Thanks to the density of $\mathscr{D}(\bar{\Omega}) \cap K_0$ in K_0 (see Lemma 1), and to the estimate (32) we obtain that K_{0h}^{nc} approaches K_0. □

Theorem 7. *We have*

$$\lim_{h \to 0} \sup_{\varepsilon \in]0,1]} \|u^\varepsilon - u_h^{\mathrm{nc},\varepsilon}\|_{1,\Omega_h} = 0.$$

This theorem shows that the $\mathbb{P}_1$ nonconforming method is robust. We now consider the approximation $u_h^{\mathrm{nc},\varepsilon}$ obtained by this method on an arbitrary mesh. In Fig. 8, one can see that $u_h^{\mathrm{nc},\varepsilon}$ matches with $u_{h,\mathrm{ref}}^\varepsilon$ in agreement with the previous mathematical study. Fig. 9 presents a convergence test for different values of ε. Those numerical experiments has been done with the C++ library GETFEM++

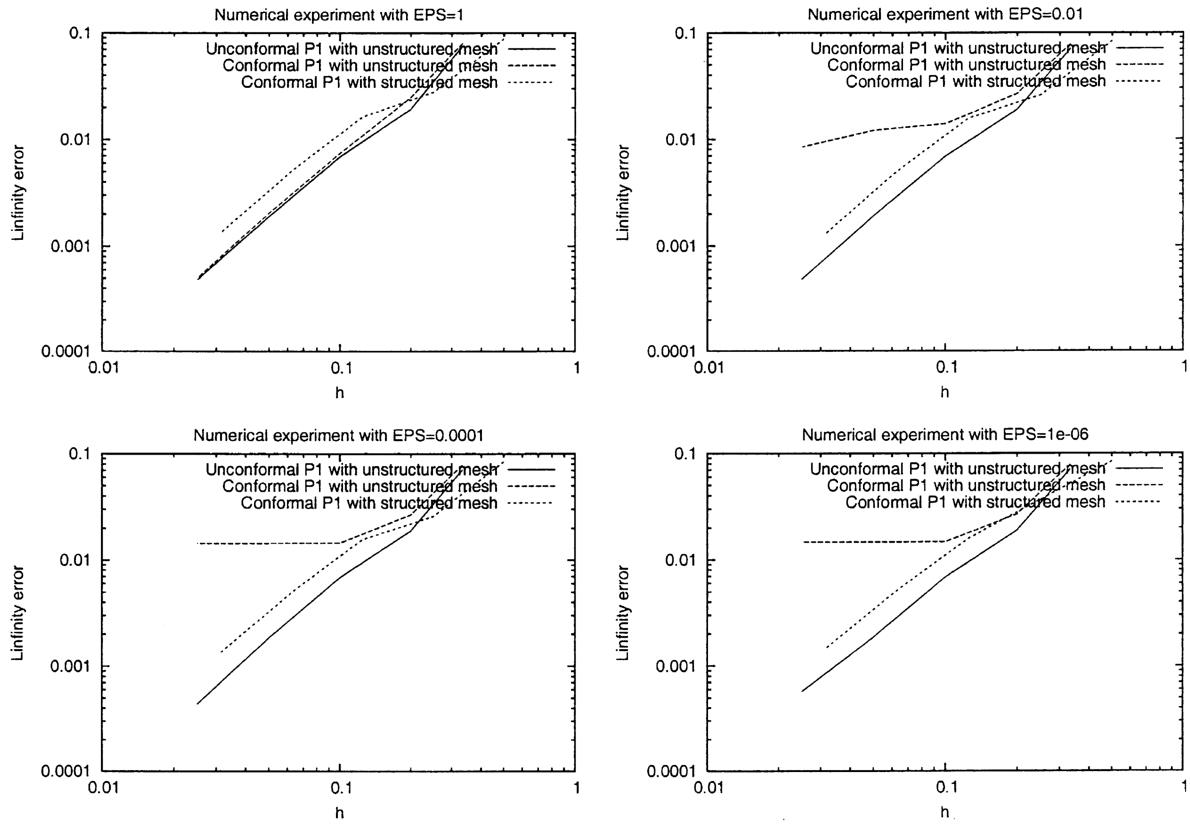

Fig. 9. Convergence test for the three methods. $L^\infty(\Omega)$ error vs. h. The $\mathbb{P}_1$ method with an unstructured mesh locks already for $\varepsilon = 10^{-2}$.

[24]. We used an Uzawa algorithm to solve the nonlinear problem. The linear step of the Uzawa algorithm has been solved thanks to a conjugate gradient with an ILUT preconditioner to prevent bad conditioning of the system. In this figure, one can see that the locking phenomenon of the $\mathbb{P}_1$ conforming method on unstructured meshes is already visible for $\varepsilon = 10^{-2}$. The $\mathbb{P}_1$ nonconforming method on unstructured meshes is even slightly better than the $\mathbb{P}_1$ conforming method on structured meshes.

5. Conclusion and prospects

In this article, we studied the numerical behavior of a class of variational inequalities in which a small parameter is involved. It consists in extending some robust methods to variational inequalities, mainly, conforming and nonconforming methods. We introduced general sufficient conditions ensuring that a numerical scheme converges uniformly with respect to the small parameter.

An example has been studied, which concerns the computation of stiff transmission problem with Signorini type conditions in anti-plane elasticity. We shown the robustness of its approximation by low-order primal methods: in the conforming case with adapted mesh, and in the nonconforming case without any restriction on the mesh.

This study shows that the robust methods used for variational equations can be extended to variational inequalities. A generalization of the mixed method for this last type of problems has been given in [4,25], where the nearly incompressible elasticity with or without friction problems have been taken as an application.

As a prospect, we intend to generalize the result in [3] which have a stronger definition of locking to the variational inequality case.

6. Uncited references

[1,20,22,26].

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